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TRAPPED MODES OF OSCILLATION OF A LIQUID FOR SURFACE-PIERCING BODIES IN OBLIQUE WAVES†

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The linear problem of the harmonic oscillations of an ideal incompressible heavy liquid with a free surface in the presence of two and more infinitely long partially submerged cylindrical bodies of arbitrary cross-section is considered. It is proved that there are configurations of the bodies which provide examples of the non-uniqueness of the boundary-value problem in the case of an arbitrary frequency of the oscillations and an arbitrary non-zero angle between the generatrix of the cylinders and the direction of propagation of the surface waves. In the case of these configurations, the homogeneous boundary-value problem has non-trivial solutions with a finite energy integral, which describe trapped modes of oscillation of the liquid.

The substantial progress in investigations of the solvability of linear problems describing the interaction of bodies with an unbounded ideal liquid with a free surface (see the review in [1], for example) is associated with the use of an inverse scheme for constructing examples of non-uniqueness which was used for the first time in [2] in the case of the plane problem of the oscillations of a liquid in the presence of partially submerged bodies. According to this scheme, a certain potential, composed of sources arranged on the free surface of the liquid and the streamlines corresponding to it, is considered. These streamlines, which begin and end in the free surface such that one of the singularities lies between the terminal points, can be chosen as the contours of bodies for which the above-mentioned potential is a solution of the homogeneous boundary-value problem.

This scheme has been used to construct examples of the non-uniqueness of the plane problem of the motion of partially submerged bodies [3]. It can also be used in the case of certain spatial problems. For instance, examples of non-uniqueness in the case of the axially symmetric problem of the oscillations of a fluid in the presence of partially submerged bodies have been constructed in [4].

The existence of trapped modes of oscillation of a liquid in the problem of the interaction of waves with partially submerged infinitely long cylindrical bodies when there is an arbitrary angle θ between the direction of propagation of the surface waves and a plane orthogonal to the ζ axis, that is, the generatrix of the system of cylindrical bodies, is investigated below (the case when $\theta = 0$ was considered previously in [2]). The existence of examples of non-uniqueness in the case of this problem has only been proved [1] for fairly small values of the angle θ. This constraint is removed below. Another family of potentials is proposed and a technique is developed which enables one to prove the existence of examples of non-uniqueness for all directions of propagation of the waves, apart from the case when propagation occurs along the generatrix of the cylinders, when $\theta = \pm\pi/2$.

1. FORMULATION OF THE PROBLEM

The problem of the oscillations of an ideal incompressible heavy liquid of finite depth with a free surface in the presence of two and more parallel, infinitely long, partially submerged cylindrical bodies is considered. The generatrix of the system of bodies ζ makes an arbitrary non-zero angle with the direction of propagation of the surface waves. It is assumed that this motion of the liquid is irrotational, harmonic in time with a frequency ω and periodic with respect to the variable ζ. The velocity potential

$$\mathrm{Re}\{u(x, y)e^{\pm ik\zeta}e^{-i\omega t}\}$$

then exists, where $k = K\sin\theta$ is the projection of the wave vector onto the ζ axis, $K = \omega^2/g$ is the wave number and g is the acceleration due to gravity.

The linear approximation of the theory of surface waves is used to determine the potential u. Here, the amplitudes of the waves and $|\nabla u|$ are assumed to be small and the boundary conditions can be

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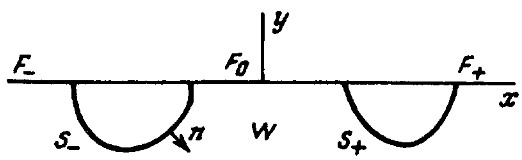

Fig. 1.

brought together in the unperturbed free surface $y = 0$. The notation in the case of two bodies is introduced in Fig. 1 where a cross-section of the plane which is orthogonal to the generatrix of the cylinders ζ is shown. W. is a section of the domain occupied by the liquid, S_+ and S_- are the wetted surfaces of the left and right bodies respectively and F_-, F_0, F_+ are three parts of the unperturbed free surface. The potential u satisfies the following boundary-value problem

$$(\nabla^2 - k^2)u = 0 \text{ in } W \tag{1.1}$$

$$Ku - u_y = 0 \text{ in } F = F_- \cup F_0 \cup F_+ \tag{1.2}$$

$$\partial u/\partial n = f \text{ in } S = S_- \cup S_+ \tag{1.3}$$

$$\nabla u \to 0 \text{ when } y \to -\infty \tag{1.4}$$

The problem of the existence of examples of non-uniqueness in the case of problem (1.1)–(1.4), that is, the existence of configurations of the bodies $S_\pm$ and potentials which provide non-trivial solutions of the problem with homogeneous condition (1.3) on the contours $S_\pm$ is investigated below. We shall henceforth therefore put $f = 0$.

Problem (1.1)–(1.4) can be regarded as the problem of finding the spectrum, where K (or k) is a characteristic number and u is an eigenfunction. If the energy functional

$$D[u] = \int_W |\nabla u|^2 dxdy + K\int_F |u|^2 dx$$

is infinite in the case of a solution u of problem (1.1)–(1.4), then it is said that the corresponding value of K belongs to the continuous spectrum of the problem. If

$$D[u] < \infty \tag{1.5}$$

then K belongs to the point spectrum and the function u is called a trapped mode of oscillation.

In the general case, in order to describe the process of the radiation and scattering of waves, problem (1.1)–(1.4) must be supplemented with conditions which determine the form of the wave motion at infinity. Here, these conditions can be omitted since only trapped modes will be considered and condition (1.5) implies that there are no waves at infinity.

The aim of the following three sections of this paper is to prove the following assertion.

Theorem. Pairs of bodies $S_\pm$, for which problem (1.1)–(1.5) has a non-trivial solution, exist for any values of the wave number K and the angle $\theta \neq \pm\pi/2$.

It will subsequently be shown in Section 5 that the proof of the theorem transfers directly to the case of more than two bodies.

2. EXAMPLES OF NON-UNIQUENESS

In order to construct potentials which provide examples of non-uniqueness, Green's function of the problem, that is, the potential of a line of sources arranged on the free surface $(x, y) = (\xi, 0)$, is required. This function can be written in the form

$$\mathrm{Re}\{G(x, y, \xi)e^{\pm ik\zeta}e^{-i\omega t}\}$$

where the potential G is the solution of the problem

$$(\nabla_{x,y}^2 - k^2)G(x, y, \xi) = 0 \text{ when } y < 0, -\infty < x < \infty \tag{2.1}$$

$$KG - G_y = 0 \text{ when } y = 0, x \neq \xi \tag{2.2}$$

$$G(x, y, \xi) \sim -\log\sqrt{(x-\xi)^2 + y^2} \text{ when } (x-\xi)^2 + y^2 \to 0 \tag{2.3}$$

$$\nabla_{x,y}G(x, y, \xi) \to 0 \text{ when } y \to -\infty \tag{2.4}$$

The function G, which satisfies relations (2.1)–(2.4), has been constructed and investigated earlier [5] and can be written in the form [1]

$$G(x, y, \xi) = 2\int_C g(t)\exp\{ly(t^2 + \mathrm{tg}^2\theta)^{1/2}\}\cos l(x-\xi)tdt \tag{2.5}$$

$$l = \sqrt{K^2 - k^2},\ g(t) = \frac{\sec\theta + (t^2 + \mathrm{tg}^2\theta)^{1/2}}{t^2 - 1}$$

where the integration is carried out along the lower edge of the cut $(0, +\infty)$. The choice of such a contour C which passes under the point $t = 1$ corresponds to waves departing to infinity in the asymptotic form of the function G.

In order to construct examples of non-uniqueness, we make use of the potential

$$\Phi(x, y) = -(2K)^{-1}[G_x(x, y, 0) - G_x(x, y, -2a)],\ a(k) = l^{-1}\pi \tag{2.6}$$

which is made up of horizontal dipoles. It is obvious that the potential (2.6) satisfies relations (1.1), (1.2) and (1.4). The asymptotic representation

$$G_x(x, y, \xi) \sim -2\pi K e^{Ky} e^{il|x-\xi|}\mathrm{sign}(x-\xi)$$

the residual term of which decays exponentially $|x - \xi| \to \infty$, was then obtained [5]. The absence of wave components in the asymptotic form of Φ at infinity is therefore a consequence of the choice of the distance between the dipoles, equal to $2a$.

We will now consider the streamlines of the potential Φ, that is, the curves with tangents which indicate the direction of the velocity vector (Φ_x, Φ_y) at the contact point. These curves are the trajectories $(x(s), y(s))$ of the system of autonomous differential equations

$$\dot{x} = \Phi_x(x(s), y(s)),\ \dot{y} = \Phi_y(x(s), y(s)) \tag{2.7}$$

where the dot denotes differentiation with respect to the parameter s.

If a trajectory of system (2.7) starts and ends in the free surface and encompasses a point at which a dipole is located (that is, the terminal points are located in the free surface on different sides of the dipole), then such a line can be chosen as one of the contours $S_\pm$ (Fig. 1). In this case, by virtue of (2.7), the potential Φ satisfies homogeneous condition (1.3) on such a contour. The proof of the theorem therefore reduces to proving the existence of solutions of system (2.7) which encompass singular points. The advantage of potential (2.6) compared with potentials consisting of sources, which have been used previously [1], lies in the fact that there are solutions of (2.7), which encompass singular points. The advantage of potential (2.6) compared with potentials consisting of sources, which have been used previously [1], lies in the fact that there are solutions of (2.7), which encompass a singular point, in as small a neighbourhood of this point as desired. This latter fact enables us to use asymptotic methods to prove the existence of trapped modes. The local asymptotic forms of the potential and its derivatives, which are required for the proof, will be obtained in Section 3 and the proof of the theorem is therefore postponed until Section 4.

As a consequence of the evenness of the potential $\Phi(x-a, y)$ with respect to the variable x, the pattern of the streamlines, which are determined by Eqs (2.7), is symmetrical about the line $\{x = -a\}$. We shall therefore henceforth confine the treatment to streamlines which are close to the right-hand dipole, which is positioned at the origin of the system of coordinates.

3. LOCAL ASYMPTOTIC FORMS OF THE POTENTIAL Φ AND ITS DERIVATIVES

We will now study the behaviour of the potential Φ in the neighbourhood of the origin of the system of coordinates. It is convenient to write $\Phi(x, y) = \Phi_0(x, y) - \Phi_0(x + 2a, y)$, where

$$\Phi_0(x,y) = \int_0^{+\infty} g_0(t,y)e^{lyt}\sin lxt dt \tag{3.1}$$

$$g_0(t,y) = t\cos\theta g(t)\exp\{ly((t^2+\mathrm{tg}^2\theta)^{1/2}-t)\}$$

Since the integrands in the representations of the functions $\Phi_0(x, y)$ and $\Phi_0(x + 2a, y)$ have the same singularity (a simple pole) at $t = 1$ and the potential Φ is the difference between these functions, the method of regularizing the integral in (3.1) is unimportant. We shall assume that this integral is evaluated in the sense of a principal value. Next, we define the function

$$g_1(t,y) = g_0(t,y) - \cos\theta - \frac{1}{t}h_0(y,\theta),\ h_0(y,\theta) = 1 + \frac{1}{2}Ky\sin^2\theta \tag{3.2}$$

such that $g_1(t, y) = \mathcal{O}(t^{-2})$ as $t \to \infty$ uniformly with respect to y when $-\infty < c \leqslant y \leqslant 0$. Representation (3.1) of the function Φ_0 can then be rewritten in the form

$$\begin{aligned}\Phi_0(x,y) = &\int_2^{+\infty} g_1(t,y)e^{lyt}\sin lxt dt + \cos\theta\int_2^{+\infty} e^{lyt}\sin lxt dt +\\ &+h_0(y,\theta)\int_2^{+\infty}\frac{e^{lyt}\sin lxt}{t}dt + \alpha(x,y)\end{aligned} \tag{3.3}$$

where

$$\alpha(x,y) = \int_0^2\left[g_0(t,y)e^{lyt}\sin lxt - \frac{e^{Ky}\sin lx}{t-1}\right]dt \tag{3.4}$$

The expression in the square brackets is a function of the parameter t, which is analytic in the interval of integration. The analytic form of this expression with respect to the pair of variables $(x, y) \in \mathbf{R}^2$ when $t \neq 1$ is also obvious. In view of the above-mentioned properties, Hartogs lemma (see [6, p. 216], for example) guarantees the analyticity of the integrand in (3.4) with respect to the set of variables $(x, y, t) \in \mathbf{R}^2 \times [0, 2]$. The analyticity of the function $\alpha(x, y)$ with respect to the set of parameters $(x, y) \in \mathbf{R}^2$ follows from this.

The estimate $g_1(x, y)$ when $t \to +\infty$ mentioned above guarantees the uniform convergence of the first integral in (3.3) in any compact subset of the lower half-plane $\overline{\mathbf{R}}^2_- = \{y \leqslant 0\}$. Hence, the first term on the right-hand side of (3.3) is a function of the parameters (x, y) which is continuous in $\overline{\mathbf{R}}^2_-$.

We now introduce the complex coordinates $z = x + iy$ and use the formula

$$\int_2^{+\infty}\frac{e^{-ilzt}}{t}dt = \mathrm{E}_1(2ilz),\ \ \mathrm{Im}\, z < 0 \tag{3.5}$$

where E_1 is an integral function of the exponent [7, 5.1.1]. It is well known that the expression $\mathrm{E}_1(z) + \log z$ is a function of the parameter z [7, 5.1.11] which is regular in $\mathbf{C}$. From (3.3) and (3.5), we thereby obtain

$$\Phi_0(x,y) = \frac{x}{K(x^2+y^2)} + \arg K(x+iy) + \phi(x,y) \tag{3.6}$$

where $\phi(x, y)$ is a function which is continuous at the point (0, 0). It follows from results which have been obtained previously [5] that the potential $\Phi_0(x, y)$ can be continued analytically into the upper half-plane with a cut along the positive part of the y axis, on which it has a discontinuity. A representation, similar to (3.6), therefore holds for the function Φ, where the contribution from the second dipole is included in the continuous term.

We will now consider the complex potential

$$w(Kz) = 1/(Kz) - i\log Kz$$

The streamlines of the potential w, which are determined by the equations $\mathrm{Im}\{w(Kz)\} = \text{const}$, are symmetrical about the y axis and join the two parts of the free surface (see Fig. 2, where the streamlines are shown for $x > 0$). Meanwhile, the expression $\mathrm{Re}\{w(Kz)\}$ is identical to the sum of the first two terms of representation (3.6). It is therefore to be expected that the trajectories which are defined by Eq. (2.7) will be close in some sense to the streamlines of the potential w and also form contours which contain the singularity. The remainder of this paper is concerned with the proof of this supposition.

Asymptotic representations of the functions Φ_x, Φ_y, Φ_{xx}, Φ_{yy}, and Φ_{xy}, when $z \to 0$ are required. For this purpose, we shall make use of the scheme employed to obtain representation (3.6). To fix our ideas, we will consider Φ_x. It follows from (3.1) that

$$\Phi_x(x,y) = l\int_2^{+\infty} tg_0(t,y)e^{lyt}\cos lxt\,dt + \sigma(x,y) \tag{3.7}$$

$$\sigma(x,y) = \alpha_x(x,y) + \frac{\partial\Phi_0}{\partial x}(x+2a,y)$$

where the function $\sigma(x,y)$ is analytic in $\mathbb{R}^2$ with a cut $\{x = -2a, y > 0\}$. Next, suppose that

$$g_2(t,y) = tg_0(t,y) - t\cos\theta - h_0(y,\theta) - \frac{1}{t}\cos\theta h_1(y,\theta)$$

$$h_1(y,\theta) = 1 + \frac{1}{2}(1+Ky)\mathrm{tg}^2\theta + \frac{1}{8}l^2y^2\mathrm{tg}^4\theta$$

In this case, $g_2(x,y) = \mathcal{O}(t^{-2})$ and $t \to +\infty$ is uniform with respect y, $-\infty < c \leqslant y \leqslant 0$. Formula (3.7) can then be written as follows:

$$l^{-1}\Phi_x(x,y) = \int_2^{+\infty} g_2(t,y)e^{lyt}\cos lxt\,dt + \frac{\sigma(x,y)}{l} + \cos\theta\int_2^{+\infty} te^{lyt}\cos lxt\,dt +$$

$$+h_0(y,\theta)\int_2^{+\infty} e^{lyt}\cos lxt\,dt + \cos\theta h_1(y,\theta)\int_2^{+\infty}\frac{e^{lyt}\cos lxt}{t}dt$$

The definition of the function g_2 ensures the uniform convergence of the first integral on the right-hand side of the last formula and, as a consequence of this, the continuity of the indicated term in $\mathbf{R}^2_-$. Then, after some reduction, we arrive at the representation when $r = |x + iy| \to 0$

$$\Phi_x = \frac{y^2 - x^2}{K(x^2+y^2)^2} - \frac{y}{x^2+y^2} - \frac{1}{2}K(1+\cos^2\theta)\log K|x+iy| + \mathcal{O}(1) \tag{3.8}$$

In a similar way, it can be shown that, as $r \to 0$

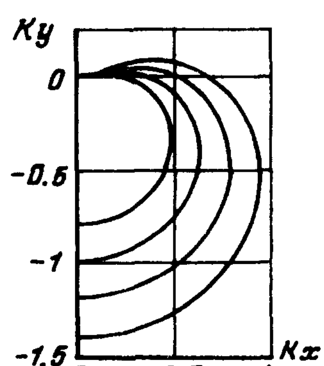

Fig. 2.

$$\Phi_y = -\frac{2xy}{K(x^2+y^2)^2} + \frac{x}{x^2+y^2} + O(1),\ \Phi_{xy} = \frac{2y(3x^2-y^2)}{K(x^2+y^2)^3} + O(r^{-2})$$
$$\Phi_{xx} = \frac{2x(x^2-3y^2)}{K(x^2+y^2)^3} + O(r^{-2}),\ \Phi_{xx} + \Phi_{yy} = O(r^{-2}) \tag{3.9}$$

4. PROOF OF THE EXISTENCE OF EXAMPLES OF NON-UNIQUENESS

Here, it will be convenient to change the parametrization of the curves, defined by system (2.7), by rewriting the system as follows:

$$\dot{x} = X(x(s), y(s)),\ \dot{y} = Y(x(s), y(s)) \tag{4.1}$$
$$X(x,y) = K(x^2+y^2)^{3/2}\Phi_x(x,y),\ Y(x,y) = K(x^2+y^2)^{3/2}\Phi_y(x,y)$$

We also consider a model system, the right-hand side of which contains the leading terms of the asymptotic forms of the functions X and Y when $r \to 0$

$$\dot{x}_0 = X_0(x_0(s), y_0(s)),\ \dot{y}_0 = Y_0(x_0(s), y_0(s)) \tag{4.2}$$
$$X_0(x,y) = (y^2-x^2)(x^2+y^2)^{-1/2},\ Y_0(x,y) = -2xy(x^2+y^2)^{-1/2}$$

The trajectories of (4.2) in $\overline{\mathbf{R}}^2_-$ are circles which we write as follows:

$$x_0(s) = \rho\sin\tau(s),\ y_0(s) = -\rho(1+\cos\tau(s)) \tag{4.3}$$

In this case, the values $\tau = \pm\pi$ correspond to the point (0, 0).

We will now study the properties of the parametrization $\tau(s)$. We substitute (4.3) into (4.2) and obtain the differential equation

$$\tau'(s) = 2\sin\frac{\pi - \tau(s)}{2} \tag{4.4}$$

Also, suppose that $\tau(0) = 0$, that is, the value of the parameter $s = 0$ corresponds to the lowest point of the circle.

It is then obvious that $\sin t < t$ when $t > 0$, and, for an arbitrary value $0 < c < 1$ in a sufficiently small neighbourhood of zero $0 < t < \delta$, we have the inequality $ct < \sin t$. On applying the comparison theorem [8, 3.3.2], we obtain, in the case of solutions of Eq. (4.4) which are sufficiently close to π, the estimate

$$\pi - c^{-1}e^{-cs} \le \tau(s) \le \pi - e^{-s}$$

It follows from this that $\tau(s) \to \pm\pi$ when $s \to \pm\infty$. Values of the parameter $s = \pm\infty$ therefore correspond to the upper point of the trajectory of the system $z = 0$, but, in this case, $\tau(s) = \pm\pi + O(e^{\mp s})$ when $s \to \pm\infty$.

We now consider the solutions of systems (4.1) and (4.2), that is, the trajectories $(x^\rho(s), y^\rho(s))$ and $(x_0^\rho(s), y_0^\rho(s))$, respectively, which pass through the point $(0, -\rho)$, to which value of the parameter $s = 0$ corresponds. It can be shown using formulae (4.1), (4.2) and (3.8)–(3.9) that the maximum value of the quantities $\dot{x}^\rho(s)$, $\dot{y}^\rho(s)$, $\dot{x}_0^\rho(s)$ and $\dot{y}_0^\rho(s)$ on the parts of the trajectories lying in the half-circle $B_{d\rho} = \{(x,y)\colon |x+iy| \le d\rho, y \le 0\}$ is of the order of $O(\rho)$ when $\rho \to 0$. It follows from this that, for any value of the parameter $s_* < \infty$, it is possible to find a $d(c) \ge 1$ such that, when $|s| \le s_*$ and $0 < \rho \le c$, the trajectories $(x^\rho(s), y^\rho(s))$ and $(x_0^\rho(s), y_0^\rho(s))$ are contained in $B_{d\rho}$. We further note that the functions $|\nabla X|$, $|\nabla Y|$, $|\nabla X_0|$ and $|\nabla Y_0|$ are bounded in the neighbourhood of the origin of the system of coordinates. We now define

$$M(\rho) = \max_{z\in B_{d\rho}}\{|\nabla X_0(z)|, |\nabla Y_0(z)|\}$$
$$\varepsilon(\rho) = \max_{z\in B_{d\rho}}\{|X(z) - X_0(z)|, |Y(z) - Y_0(z)|\}$$

The comparison theorem [8, T. 3.3.1] and the estimate $\varepsilon(\rho) = \mathcal{O}(\rho^2)$ when $\rho \to 0$ then guarantee that

$$\max\{\|\mathbf{x}^{\rho}(s) - \mathbf{x}_0^{\rho}(s)\| : |s| \le s_*\} \le \frac{\varepsilon(\rho)}{M(\rho)}(e^{M(\rho)s_*} - 1) = \mathcal{O}(\rho^2) \tag{4.5}$$

We now consider a certain neighbourhood of the origin of the system of coordinates $V \subset \overline{\mathbf{R}}^2_-$ which includes the parts of the free surface to the right and to the left of the dipole. Suppose the neighbourhood V is sufficiently small such that, in it, there are no singular points of system (4.1) $X = Y = 0$, which differ from (0, 0) (the existence of such a neighbourhood follows from (3.8) and (3.9)). Now, consider a domain $S \subset V$ consisting of the solution of (4.1) which passes through the negative part of the y axis. We denote the subsets $V\backslash S$, which include the parts of the free surface when $\pm x > 0$ by $\mathcal{F}_{\pm}$, respectively. The assumption that S does not have common points with the free surface to the left of the dipole (Fig. 3) adjacent to the origin of the system of coordinates leads to a contradiction.

Actually, we note that the points $(x_0^{\rho}(s_*), y_0^{\rho}(s_*))$ lie on a straight line which passes through the origin at a certain angle which can be made as small as desired at the price of an increase in the magnitude of s_*. In view of (4.5), the curve γ_* therefore touches the free surface at the point (0, 0). We will denote the function defining γ_* in the neighbourhood of the origin by $\gamma_*(x)$. Since $\gamma_*(x) = o(x)$ when $x \to 0$, it follows from (4.1), (3.8) and (3.9) that $\gamma'_*(x)$ satisfies the equation

$$y_*'(x) = -Kx + 2x^{-1}y_*(x) + \lambda(x),\ \lambda(x) = o(x)$$

We write $y_*(x) = -Kx^2\log K\,|x| + \varphi(x)$. Then, φ satisfies the equation

$$\varphi'(x) - 2x^{-1}\varphi(x) = \lambda(x)$$

the solution of which can be represented in the form

$$\varphi(x) = x^2\Big(\text{const} - \int_x^1 \xi^{-2}\lambda(\xi)d\xi\Big) \tag{4.6}$$

Next, on taking account of the estimate $\lambda = o(x)$ when $x \to 0$, we obtain

$$\lim_{x\to 0}\int_x^1 \xi^{-2}\lambda(\xi)d\xi \Big/ \int_x^1 \xi^{-1}d\xi = \lim_{x\to 0} x^{-1}\lambda(x) = 0$$

The estimate $\varphi(x) = o(x^2\log x)$ follows from the last formula and the equality (4.6), and it is thereby established that

$$y_*(x) = -Kx^2\log K|x| + o(x^2\log x) \text{ when } x \to 0$$

This last equation contradicts the assumption that S does not have common points with the free surface to the left of the dipole, since it forbids line γ_* departing from the point (0, 0) into the lower half-plane.

Hence, the set S includes a certain part of the free surface and the dipole is located within this range. It is therefore possible to find a family of trajectories $S_0 \subset S$, the terminal points of which are located in the free surface on different sides of the point at which the dipole is situated, and this holds for any value of the angle $\theta \ne \pm\pi/2$. Since the part of the free surface which belongs to V is not a characteristic line of (4.1), the trajectories of the family S_0 touch the free surface at a non-zero angle and form contours

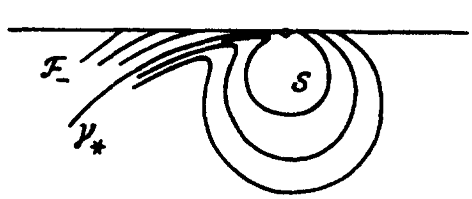

Fig. 3.

possessing the properties required to regard them as contours of the bodies $S_{\pm}$ for which Φ is a solution of problem (1.1)–(1.5). The theorem is thereby proved.

5. OTHER EXAMPLES OF NON-UNIQUENESS

We will now consider the family of potentials

$$\Psi_n^{(\pm)}(x,y)=\Phi_0(x-a_n^{(\pm)},y)\mp\Phi_0(x+a_n^{(\pm)},y)$$
$$a_n^{(\pm)}=l^{-1}(n-1/4\pm 1/4)\pi, n=1,2,\ldots$$

Here, $\Phi(x-a_1^{(+)},y)=\Psi_1^{(+)}(x,y)$ and $\Psi_n^{(+)}(\Psi_1^{(-)})$ are even (odd) functions of the variable x. The potentials $\Psi_n^{\pm}$ are wave-free and also provide examples of the non-uniqueness of problem (1.1)–(1.5). The proof presented above of the existence of streamlines which encompass singularities remains valid by virtue of its local character.

The results obtained transfer to the case when the number of bodies N, for which the eigenvalue problem (1.1)–(1.5) is formulated, is greater than two. We denote the integral part of the number $N/2$ by m. Then, for example, the potential

$$2(N-2m)[\Phi_0(x,y)+\Phi_0(x+a_1^{(+)},y)]+\sum_{n=1}^{m}\Psi_n^{(+)}(x,y)$$

is wave-free and has N singularities in the free surface.

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